1.3 Likelihoods for Regression Models

We will start with linear regression and then talk about more general models.

↓ nonlinear GLM

1.3.1 Linear Model

Consider the familiar linear model

$$Y_i = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \epsilon_i, \qquad i = 1, \ldots, n,$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are known nonrandom vectors.

$$E[\varepsilon_i] = 0 \quad \text{and} \quad Var[\varepsilon_i] = \sigma^2$$

often estimate β by $\hat{\beta}_{OLS}$, which does not require a distribution for ε_i.

For likelihood-based estimation, we need a distribution for ε_i! Start w/ $\varepsilon_i \sim N(0, \sigma^2)$.

$$\Rightarrow L(\boldsymbol{\beta}, \sigma | \{Y_i, \boldsymbol{x}_i\}_{i=1}^n) = \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}\sigma}\right) \exp\left(-\frac{(Y_i - \underset{\sim}{x}_i^\top \underset{\sim}{\beta})^2}{2\sigma^2}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (Y_i - \underset{\sim}{x}_i^\top \underset{\sim}{\beta})^2\right)$$

take log, derivatives, set = 0, solve

$$\hat{\underset{\sim}{\beta}}_{MLE} = (X^\top X)^{-1} X^\top \underset{\sim}{Y} \qquad \text{same as } \hat{\beta}_{OLS}!$$

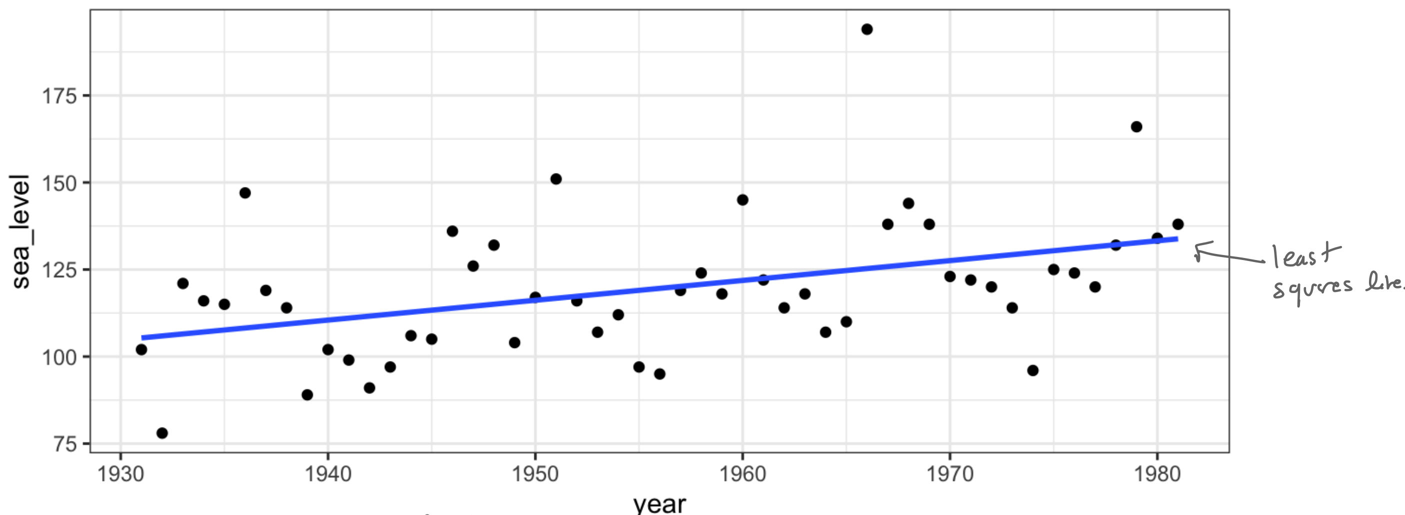
$$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^n (Y_i - \underset{\sim}{x}_i^\top \underset{\sim}{\beta})^2 \qquad \text{(only asymptotically unbiased).}$$

What do you do when ϵ_i are not Gaussian?

- transform data so ε_i look Gaussian.
- Use a different distribution for ε_i!

Example (Venice sea levels): The annual maximum sea levels in Venice for 1931–1981 are :

We know maxima are not Gaussian!



Approach 1: OLS $E[\varepsilon_i]=0$, $Var[\varepsilon_i]=\sigma^2$ No distributional assumption.

Approach 2: Assume $\varepsilon_i \sim$ Gumbel (extreme value dsn), use ML

$$f(y;\sigma)=\frac{1}{\sigma}\exp\left(-\frac{y}{\sigma}\right)\exp\left(-\exp\left(-\frac{y}{\sigma}\right)\right).$$

$$\Rightarrow L(\beta,\sigma\mid\{y_i,\underline{x}_i\}_{i=1}^n)=\prod_{i=1}^n f(y_i-\underline{x}_i^T\beta)=\prod_{i=1}^n\frac{1}{\sigma}\exp\left(-\frac{y_i-\underline{x}_i^T\beta}{\sigma}\right)\exp\left(-\exp\left(-\frac{y_i-\underline{x}_i^T\beta}{\sigma}\right)\right)$$

YOUR TURN: Fit both approaches to the venice data.

OLS

$\hat\beta_0=104.8$ $\hat\beta_1=.56$ (SE .177)

MLE, GUMBEL

$\hat\beta_0=96.8$, $\hat\beta_1=0.56$ (SE .136)

OLS vs MLE? If EV model is correct, more efficient (note: standard errors).

β_0 difference: $\overset{\text{Gumbel}}{E}[\varepsilon_i]=0.577\sigma = 0.577\hat\sigma_{MLE}=0.577(14.5)\neq 0$

$96.8+.577(14.5)=105.1$

1.3.2 Additive Errors Nonlinear Model

Previous example had 1 linear trend, 2 Non-Gaussian errors.

Non-linear additive model:

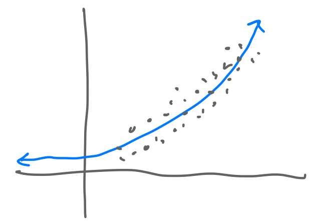
$$Y_i = g(\underline{x}_i, \underline{\beta}) + \varepsilon_i$$

Often interested in $\varepsilon_i \sim N(0, \sigma^2)$ but $g(\underline{x}_i, \underline{\beta}) \neq \underline{x}_i^T \underline{\beta}$ $\Rightarrow$ ML required.

1 non-linear trend, 2 Gaussian errors.

Example: exponential growth model

$$g(x, \underline{\beta}) = \beta_0 \exp(\beta_1 x)$$



1.3.3 Generalized Linear Models

Imagine an experiment where individual mosquitos are given some dosage of pesticide. The response is whether the mosquito lives or dies. The data might look something like:

Goal: Model the relationship between the predictor and response.

Question: What would a curve of best fit look like?

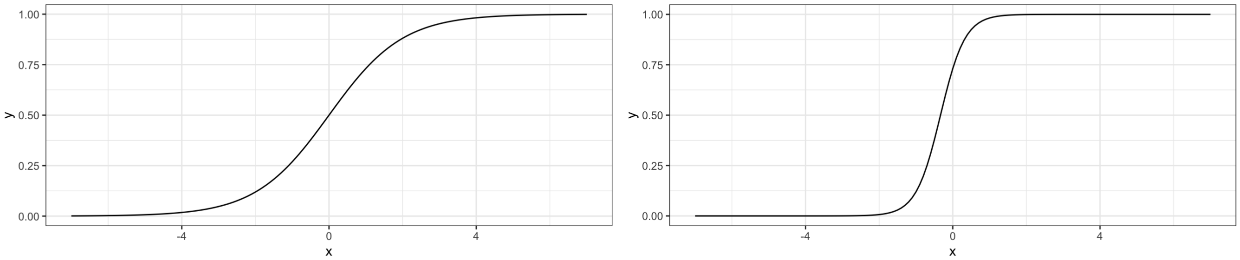
Refined Goal:

Let's build a sensible model.

Step 1: Find a function that behaves the way we want.

```r
# understanding the logistic function
# first, theta just equals x
x <- seq(-7, 7, .1)
theta <- x
y <- exp(theta)/(1 + exp(theta))
ggplot() + geom_line(aes(x, y))

# now, let theta be a linear function of x
theta <- 1 + 3*x
y <- exp(theta)/(1 + exp(theta))
ggplot() + geom_line(aes(x, y))
```



Step 2: Build a stochastic mechanism to relate to a binary response.

Step 3: Put Step 1 and Step 2 together.

Fitting our model: Does OLS make sense?

Consider the likelihood contribution.

$$L_i(p_i|Y_i) =$$

So the log-likelihood contribution is

$$\ell_i(p_i) =$$

Recall, we said $p_i = \frac{\exp(\theta_i)}{1+\exp(\theta_i)}$ was sensible.

Which gives us,

$$\ell_i(\theta_i) =$$

So the log-likelihood is

$$\ell(\theta_1, \ldots, \theta_n) =$$

To optimize?

```r
## data on credit default
data("Default", package = "ISLR")
head(Default)
```

```
##   default student   balance    income
## 1      No      No  729.5265 44361.625
## 2      No     Yes  817.1804 12106.135
## 3      No      No 1073.5492 31767.139
## 4      No      No  529.2506 35704.494
## 5      No      No  785.6559 38463.496
## 6      No     Yes  919.5885  7491.559
```

```r
## fit model with ML
m0 <- glm(default ~ balance, data = Default, family = binomial)
tidy(m0) |> kable()
```

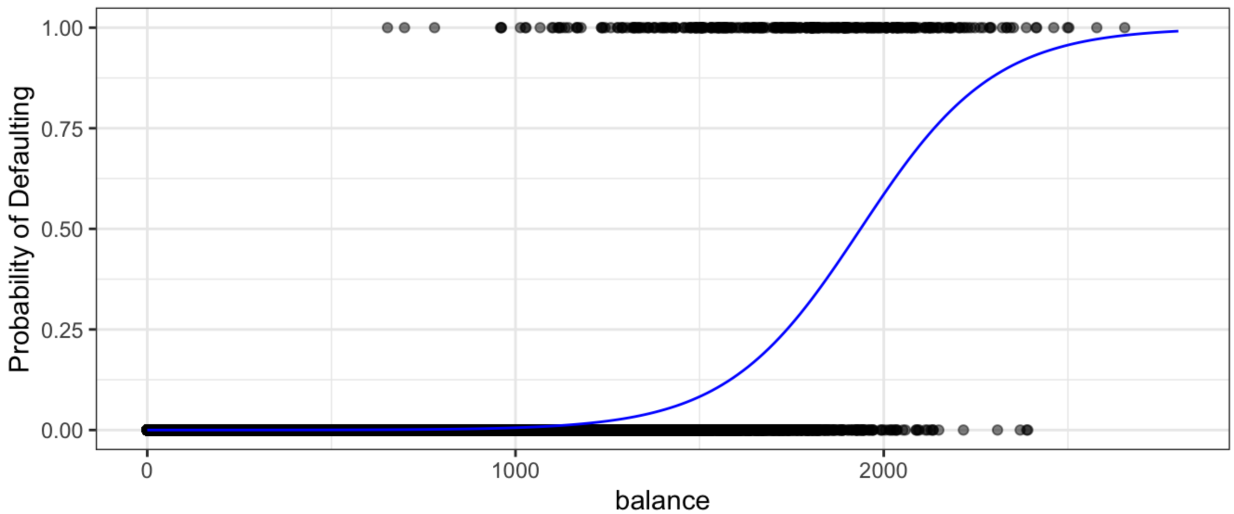
term	estimate	std.error	statistic	p.value
(Intercept)	-10.6513306	0.3611574	-29.49221	0
balance	0.0054989	0.0002204	24.95309	0

```r
glance(m0) |> kable()
```

null.deviance	df.null	logLik	AIC	BIC	deviance	df.residual	nobs
2920.65	9999	-798.2258	1600.452	1614.872	1596.452	9998	10000

```r
## plot the curve
x_new <- seq(0, 2800, length.out = 200)
theta <- m0$coefficients[1] + m0$coefficients[2]*x_new
p_hat <- exp(theta)/(1 + exp(theta))

ggplot() +
  geom_point(aes(balance, as.numeric(default) - 1), alpha = 0.5, data
        = Default) +
  geom_line(aes(x_new, p_hat), colour = "blue") +
  ylab("Probability of Defaulting")
```


Probability of Defaulting
1.00
0.75
0.50
0.25
0.00
0
1000
2000
balance

In general, a GLM is three pieces:

1. The random component

2. The systemic component

3. A linear predictor

Remarks:

Example (Poisson regression):

Consider a general family of distributions:

$$\log f(y_i; \theta_i, \phi) = \frac{y_i\theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i, \phi).$$

Example (Normal model):

We can learn something about this distribution by considering it's mean and variance. Because we don't have an explicit form of the density, we rely on two facts:

1. $\mathrm{E}\left[\frac{\partial \log f(Y_i;\theta_i,\phi)}{\partial \theta_i}\right] = 0.$

2. $\mathrm{E}\left[\frac{\partial^2 \log f(Y_i;\theta_i,\phi)}{\partial \theta_i^2}\right] + \mathrm{E}\left[\left(\frac{\partial \log f(Y_i;\theta_i,\phi)}{\partial \theta_i}\right)^2\right] = 0.$

For $\log f(y_i;\theta_i,\phi) = \frac{y_i\theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i,\phi)$,

Example (Bernoulli model):

$$f(y_i; p_i) = p_i^{y_i}(1 - p_i)^{1-y_i}$$

Finally, back to modelling. Our **goal** is to build a relationship between the mean of Y_i and covariates $\boldsymbol{x}_i$.

Example (Bernoulli model, cont'd):

1.4 Marginal and Conditional Likelihoods

Consider a model which has $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, where $\boldsymbol{\theta}_1$ are the parameters of interest and $\boldsymbol{\theta}_2$ are nuisance parameters.

One way to improve estimation for $\boldsymbol{\theta}_1$ is to find a one-to-one transformation of the data $\boldsymbol{Y}$ to $(\boldsymbol{V}, \boldsymbol{W})$ such that either

The key feature is that one component of each contains only the parameter of interest.

Example (Neyman-Scott problem): Let $Y_{ij}, i = 1, \dots, n, j = 1, 2$ be intependent normal random variables with possible different means μ_i but the same variance σ^2.

Our goal is to estimate σ^2. Should we be able to?

Following the usual arguments,

$$\begin{aligned}
\hat{\mu}_{i,\text{MLE}} &= \frac{Y_{i1} + Y_{i2}}{2} \\
\hat{\sigma}^2_{\text{MLE}} &= \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{2} (Y_{ij} - \hat{\mu}_{i,\text{MLE}})^2
\end{aligned}$$

$$\mathrm{E}[\hat{\sigma}^2_{\mathrm{MLE}}] =$$

A reworking of the data seems more promising. Let,

$$V_i = \frac{Y_{i1} - Y_{i2}}{\sqrt{2}} \qquad \text{and} \qquad W_i = \frac{Y_{i1} + Y_{i2}}{\sqrt{2}}$$

For conditional likelihoods, we can often exploit the existence of sufficient statistics for the nuisance parameters under the assumption that the parameter of interest is known.

Example (Exponential Families): The structure of exponential families is such that it is often possible to exploit their properties to eliminated nuisance parameters. Let Y have a density of the form

$$f(y;\boldsymbol{\eta}) = h(y)\exp\left\{\sum_{i=1}^{s}\eta_i T_i(y) - A(\boldsymbol{\eta})\right\},$$

then

Thus, exponential families often provide an automatic procedure for finding $\boldsymbol{W}$ and $\boldsymbol{U}$.

Example (Logistic Regression): For binary Y_i, the standard logistics regression model is

$$P(Y_i = 1) = p_i(\boldsymbol{x}_i, \boldsymbol{\beta}) = \frac{\exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})}{1 + \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})}$$

and the likelihood is

$$L(\boldsymbol{\beta} | \boldsymbol{Y}, \boldsymbol{X}) =$$