2 Profile Likelihood

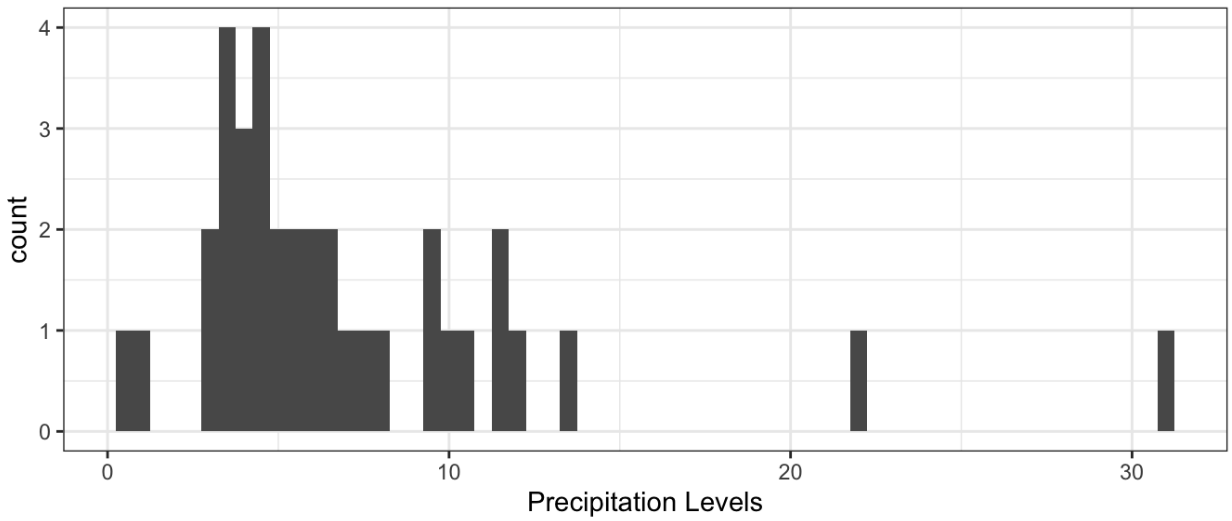
The term "profile likelihood" can mean multiple things.

2.1 Analytical Methods via Profile Likelihoods

In certain problems it is possible to maximize the log likelihood for part of $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$ without actually knowing the value of the other part.

The **profile likelihood** is the usual likelihood with the known function of part of the parameter vector inserted for that parameter, making the likelihood only a function of one part of the vector.

Example (Hurricane Data, Cont'd): For 36 hurricanes that had moved far inland on the East Coast of the US in 1900-1969, maximum 24-hour precipitation levels during the time they were over mountains.



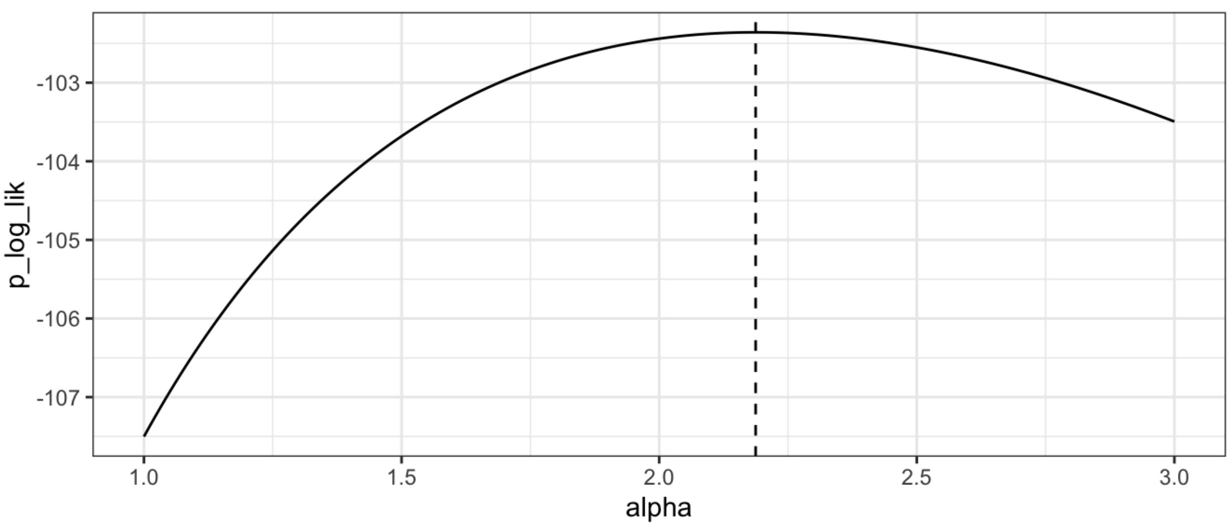
We modeled the precipitation levels with a gamma distribution, which has log likelihood

$$\ell(\alpha, \beta) = -n \log \Gamma(\alpha) - n\alpha \log \beta + (\alpha - 1) \sum \log Y_i - \frac{\sum Y_i}{\beta}$$

```
gamma_prof_loglik <- function(alpha, data) {
  beta <- mean(data) / alpha
  sum(dgamma(data, alpha, scale = beta, log = TRUE))
}

## get maximum profile likelihood estimate
alpha_mple <- optim(1, gamma_prof_loglik, data = hurr_rain, method =
        "BFGS", control = list(fnscale = -1))

## plot profile likelihood
data.frame(alpha = seq(1, 3, length.out = 200)) |>
  rowwise() |>
  mutate(p_log_lik = gamma_prof_loglik(alpha, hurr_rain)) |>
  ggplot() +
  geom_line(aes(alpha, p_log_lik)) +
  geom_vline(aes(xintercept = alpha_mple$par), lty = 2)
```



2.2 Numerical Methods via Profile Likelihoods

The log likelihood can be maximized over one portion of the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$ for any fixed value of the other, even if that maximization cannot be expressed as an explicit function.

We can define a profile likelihood as

$$L^p(\boldsymbol{\theta}_2) = \max_{\boldsymbol{\theta}_1} L(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2).$$

The profile likelihood and log profile likelihood behave in many ways like true likelihood functions:

1. The estimate of $\boldsymbol{\theta}_2$ found by maximizing $L^p(\boldsymbol{\theta}_2)$ is the MLE of $\boldsymbol{\theta}_2$.

2. A likelihood ratio test statistics formed with the profile likelihood has a limiting χ^2 distribution.

3. A profile likelihood confidence region is a valid approximate confidence region for $\boldsymbol{\theta}_2$.

Where does this confidence region come from?

However, these are *not* full likelihood functions.