"Misspecified Models" "M-estimation"

Estimating Equations

Now we will consider "robustifying" inference so that misspecification does not invalidate our resulting inference.

Motivating **Example:** Consider the $\boldsymbol{Z} = (Z_1, \ldots, Z_5)^\top$ with cdf

$$F(\boldsymbol{z}; \alpha) = \exp\left\{ -\left(z_1^{-\frac{1}{\alpha}} + z_2^{-\frac{1}{\alpha}} + z_3^{-\frac{1}{\alpha}} + z_4^{-\frac{1}{\alpha}} + z_5^{-\frac{1}{\alpha}} \right)^{\alpha} \right\}, \quad \boldsymbol{z} \geq \boldsymbol{0}, \alpha \in (0, 1].$$

If $\alpha = 1$ independence

$\alpha \to 0$ complete dependence ($Z_i = Z_j$ w.p. 1).

Marginal:

$P(Z_i \le z) = \exp\left[-(z^{-1/\alpha})^{\alpha}\right] = \exp(-z^{-1})$

"Unit Frechet"

Comments:

1. F is "max-stable." → suitable for multivariate extreme value data

defn $[F(nz)]^n = F(z)$

$$\begin{aligned}[F(nz)]^n &= \left(\exp\left[-\{(nz_1)^{-1/\alpha} + \ldots + (nz_5)^{-1/\alpha}\}^{\alpha}\right]\right)^n \\ &= \left(\exp\left[-\{n^{-1/\alpha}(z_1^{-1/\alpha} + \ldots + z_5^{-1/\alpha})\}^{\alpha}\right]\right)^n \\ &= \left(\exp\left[-n^{-1}(z_1^{-1/\alpha} + \ldots + z_5^{-1/\alpha})^{\alpha}\right]\right)^n \\ &= \exp\left[-(z_1^{-1/\alpha} + \ldots + z_5^{-1/\alpha})^{\alpha}\right] \quad //\end{aligned}$$

2. $Z_1, \ldots, Z_5$ are exchangeable. order doesn't matter

$P(Z_1, \ldots, Z_5) = P(Z_3, Z_2, Z_4, Z_5, Z_1)$ etc.

Realistic? Maybe not.

But this gives us equal pairwise dependence ⇒ which can help reduce # parameters.

↳ and illustrate the concept of an estimating equation.

Let's consider the likelihood.

Suppose we observe $\underline{z}_i = (z_{i1}, \dots, z_{i5})^T$, $i = 1, \dots, n$ iid $\sim F$. We want to estimate α.

We need to find the density, i.e. $\frac{\partial^5 F}{\partial z_1 \cdots \partial z_5}$

$$\frac{\partial F}{\partial z_1} = \exp\left[-\left(z_1^{-1/\alpha} + \dots + z_5^{-1/\alpha}\right)^{\alpha}\right] \times \left\{-\alpha\left(z_1^{-1/\alpha} + \dots + z_5^{-1/\alpha}\right)^{\alpha-1}\right\} \times \left\{-\frac{1}{\alpha} z_1^{-\frac{1}{\alpha}-1}\right\}$$

$$\frac{\partial^2 F}{\partial z_1 \partial z_2} \overset{\text{product rule}}{=} \exp\left[-\left(z_1^{-1/\alpha} + \dots + z_5^{-1/\alpha}\right)^{\alpha}\right] \times \left\{-\alpha\left(z_1^{-1/\alpha} + \dots + z_5^{-1/\alpha}\right)^{\alpha-1}\right\}^2 \cdot \times \left\{-\frac{1}{\alpha} z_2^{-\frac{1}{\alpha}-1}\right\} \cdot \times \left\{-\frac{1}{\alpha} z_1^{-\frac{1}{\alpha}-1}\right\}$$
$$+ \exp\left[-\left(z_1^{-1/\alpha} + \dots + z_5^{-1/\alpha}\right)^{\alpha}\right] \times \left\{-\alpha(\alpha-1)\left(z_1^{-1/\alpha} + \dots + z_5^{-1/\alpha}\right)^{\alpha-2}\right\} \cdot \times \left\{-\frac{1}{\alpha} z_2^{-\frac{1}{\alpha}-1}\right\} \cdot \times \left\{-\frac{1}{\alpha} z_1^{-\frac{1}{\alpha}-1}\right\}$$

$\frac{\partial^3 F}{\partial z_1 \partial z_2 \partial z_3}$ = product rule on each of the 2 terms $\longrightarrow$ 4 terms.

by the time we get to $\frac{\partial^5 F}{\partial z_1 \cdots \partial z_5}$ things are gross just to write the likelihood!

How about if we were to just use pairs of points to estimate α?

$$F_{Z_1 Z_2}(z_1, z_2) = \exp\left[-\left(z_1^{-1/\alpha} + z_2^{-1/\alpha}\right)^{\alpha}\right]$$

$$\frac{\partial^2 F}{\partial z_1 \partial z_2} = \exp\left[-\left(z_1^{-1/\alpha} + z_2^{-1/\alpha}\right)^{\alpha}\right] (z_1 z_2)^{-\frac{1}{\alpha}-1} \left\{\left(\frac{1}{\alpha}-1\right)\left(z_1^{-1/\alpha} + z_2^{-1/\alpha}\right)^{\alpha-2} + \left(z_1^{-1/\alpha} + z_2^{-1/\alpha}\right)^{2\alpha-2}\right\}.$$

If we just used $(z_{1i}, z_{2i}), i = 1, \ldots, n$ would the likelihood based on the bivariate density be a good estimator for α?

Yes: unbiased

No: inefficient (not using all data).

What if we took all $\binom{5}{2} = 10$ pairs? (z_{1i}, z_{2i}), (z_{1i}, z_{3i}), ...

Yes: unbiased, efficient (using all data).

No: It's not the right likelihood!

Composite likelihood.

Let's try it.

```r
library(evd)
# simulate data with alpha = 0.5
alpha <- 0.5
z <- rmvevd(500, dep = alpha, d = 5, mar = c(1, 1, 1))

## bivariate density
d_bivar <- function(z, alpha){
    #here "z" is a single observation (ordered pair)
    inside <- z[1]^(-1/alpha) + z[2]^(-1/alpha)
    one <- exp(-inside^alpha)
    two <- (z[1]*z[2])^(-1 / alpha - 1)
    three <- (1 / alpha - 1)*inside^(alpha - 2)
    four <- inside^(2 * alpha - 2)
    one*two*(three + four)
}

d_bivar(c(4, 5), alpha = alpha)
```

```
## [1] 0.003650963
```

```r
dmvevd(c(4,5), dep = alpha, d = 2, mar = c(1,1,1))
```

```
## [1] 0.003650963
```

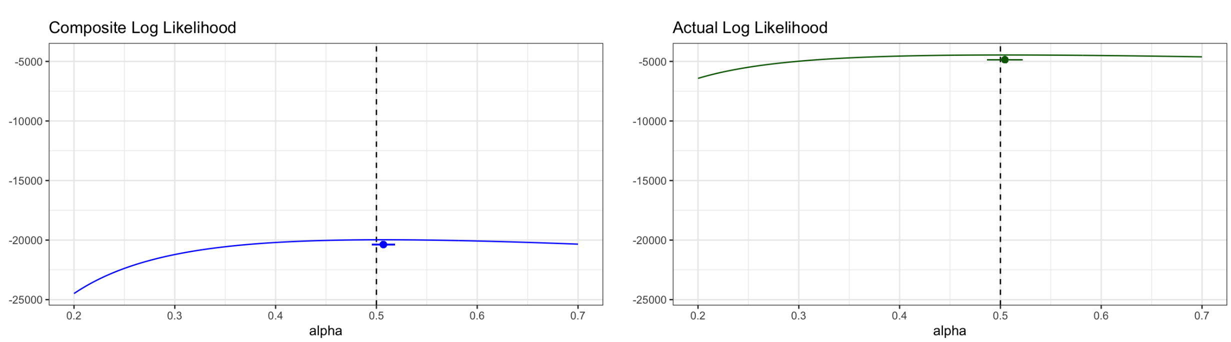
```r
## estimate alpha
log_pair_lhood <- function(alpha, z) {
    #here "z" is bivariate matrix of observations
    inside <- z[, 1]^(-1 / alpha) + z[, 2]^(-1 / alpha)
    log_one <- -inside^alpha
    log_two <- (-1 / alpha - 1) * (log(z[, 1]) + log(z[, 2]))
    three <- (1 / alpha - 1) * inside^(alpha - 2)
    four <- inside^(2 * alpha - 2)
    contrib <- log_one + log_two + log(three + four)
    return(sum(contrib))
}

all_pairs_lhood <- function(alpha, z) {
```

```r
    expand.grid(dim1 = seq_len(ncol(z)),
                dim2 = seq_len(ncol(z))) |>
      filter(dim1 < dim2) |>       rowwise() |>
      mutate(log_pair_lhood = log_pair_lhood(alpha, cbind(z[, dim1],
        z[, dim2]))) |>
    ungroup() |>      summarise(res = sum(log_pair_lhood)) |>
      pull(res)}
alpha_mple <- optim(.2, lower = .01, upper = .99, all_pairs_lhood, z
        = z, method = "Brent", hessian = TRUE, control =
        list(fnscale = -1))
(ci_mple <- alpha_mple$par + c(-1.96, 1.96)*sqrt(-1 /
        alpha_mple$hessian[1, 1]))
```

```
## [1] 0.4954979 0.5182678
```

```r
## checking coverage
#checking coverage
B <- 200
coverage <- numeric(B)
for(k in seq_len(B)) {
    z_k <- rmvevd(500, dep = .5, d = 5, mar = c(1, 1, 1))
    alpha_mple_k <- optim(.2, lower = .01, upper = .99,
        all_pairs_lhood, z = z_k, method = "Brent", hessian = TRUE,
        control = list(fnscale = -1))
    ci <- alpha_mple_k$par + c(-1.96, 1.96)*sqrt(-1 /
        alpha_mple_k$hessian[1, 1])
    coverage[k] <- as.numeric(ci[1] < alpha & ci[2] > alpha)
}
mean(coverage)
```

```
## [1] 0.745
```



So, it looks like the point estimate from the pairwise likelihood is ok, but we need to be able to get an appropriate measure of uncertainty.

The proper adjustment is

1 Introduction

M-estimators are solutions of the vector equation

$$\sum_{i=1}^{n} \boldsymbol{\psi}(\mathbf{Y}_i, \boldsymbol{\theta}) = \mathbf{0}.$$

In the likelihood setting, what is $\boldsymbol{\psi}$?

Example: Let $Y_1, \ldots, Y_n$ be independent, univariate random variables. Is $\theta = \overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$ an M-estimator?

Example: Consider the mean deviation from the sample mean,

$$\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} |Y_i - \overline{Y}|.$$

Is this an M-estimator?