2 Basic Approach

M-estimators are solutions of the vector equation

$$\sum_{i=1}^{n} \boldsymbol{\psi}(\boldsymbol{Y}_i, \boldsymbol{\theta}) = \mathbf{0}.$$

but what are they estimating?

Example (Sample Mean, cont'd): Recall we said $\theta = \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is an M-estimator for $\psi(Y_i, \theta) = Y_i - \theta$. What is the true parameter?

To arrive at the sandwich estimator, assume $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n \overset{iid}{\sim} F$ and define

$$\boldsymbol{G}_n(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\psi}(\boldsymbol{Y}_i; \boldsymbol{\theta}).$$

Taylor expansion of $\boldsymbol{G}_n(\boldsymbol{\theta})$ around $\boldsymbol{\theta}_0$ evaluated at $\hat{\boldsymbol{\theta}}$ yields

Define $\boldsymbol{A}(\boldsymbol{\theta}_0) = \mathrm{E}_F[-\boldsymbol{\psi}'(\boldsymbol{Y}_1, \boldsymbol{\theta}_0)]$.

2.1 Estimators for $\boldsymbol{A}, \boldsymbol{B}$

If the data truly come from the assumed parametric family $f(y; \boldsymbol{\theta})$,

One of the key contributions of M-estimation theory is to point out what happens when the assumed parametric family is not correct.

We can use empirical estimators of $\boldsymbol{A}$ and $\boldsymbol{B}$:

Example (Coefficient of Variation): Let $Y_1, \ldots, Y_n$ be idd from some distribution with finite fourth moment. The coefficient of variation is defined at $\hat{\theta}_3 = s_n/\overline{Y}$.

Define a three dimensional $\boldsymbol{\psi}$ so that $\hat{\theta}_3$ is defined by summing the third component. What is the vector valued function $\boldsymbol{\psi}$ which yields an M-estimator for the coefficient of variation?

What parameter vector is being estimated by the M-estimator?

What are the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$?

Write out the asymptotic variance, $\boldsymbol{V}$.

Assume Y_i are iid from a normal distribution with mean 10 and standard deviation 1. Calculate $V_{3,3}$. Assume you have a same of size 25 and you get an estimated coefficient of variation of 0.11. Give the asymptotic 95% confidence interval.