Bootstrap Methods

Typically we use (asymptotic) theory to derive the sampling distribution of a statistic. From the sampling distribution, we can obtain the variance, construct confidence intervals, perform hypothesis tests, and more.

Challenge:

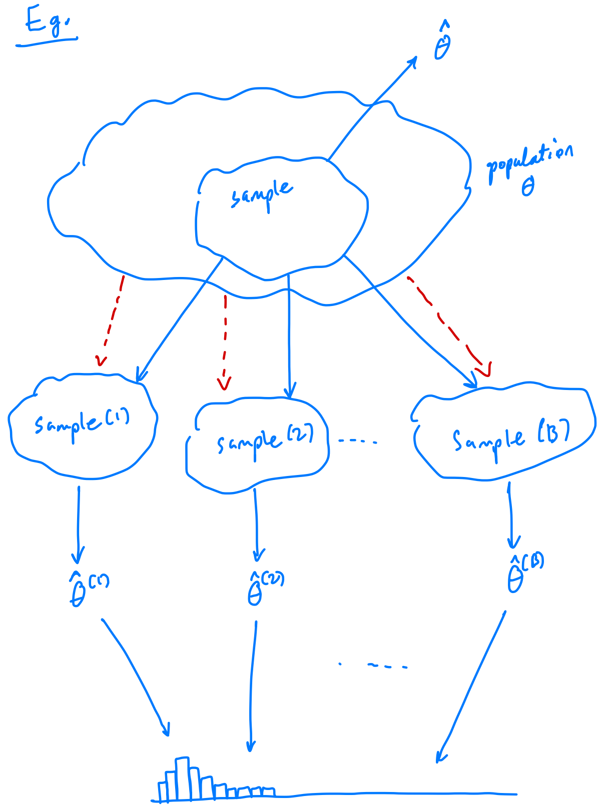
what if the sampling is impossible to obtain or asymptotic theory doesn't hold?

Basic idea of bootstrapping:

Use the data to approximate the sampling distribution of the statistic.

How? Estimate the sampling distribution by creating a large # of datasets that we might have seen and compute the statistic on each of these data sets.

Eg.



Estimated sampling dsn of $\hat{\theta}$

Goals:

estimate bias, se, CI's when

1. There is doubt about whether distributional assumptions are met.
2. There is doubt about whether asymptotic results are valid.
3. Theory to derive dsn is too hard.

In reality, we only have a sample, need to make sample$^{(1)}$, ..., sample$^{(B)}$

"Bootstrap World" where the data analyst knows everything.

idea: treat the sample $Y_1, \dots, Y_n$ as the population.

E.g., we are interested in the variance of an estimator

- In "bootstrap world" we can calculate the exact variance b/c we have access to the "population"
- In practice, estimate variance by repeatedly sampling from the pseudo-population.

Real world

True population $Y_1, Y_2, \dots$ w/ dsn F_θ

True pop. parameter θ

$\downarrow$ sample

$Y_1, \dots, Y_n \Rightarrow \hat{\theta}(Y_1, \dots, Y_n)$ is estimator

$MSE = E_F\left[(\hat{\theta} - \theta)^2\right]$

If we don't have access to F (we don't), we can't take this expectation.

If we had access to the population, (we don't!) could estimate MSE w/

$$\hat{MSE} = \frac{1}{rep} \sum_{i=1}^{rep} (\hat{\theta}_i - \theta)^2$$

Bootstrap world

$Y_1, \dots, Y_n$ is population

$\hat{\theta}$ is true value of parameter

Since we have access to the population,

$$\hat{MSE}_{BOOT} = \frac{1}{\text{Boot Rep}} \sum_{i=1}^{\text{Boot Rep}} (\hat{\theta}_i - \hat{\theta})^2$$

We hope $\hat{MSE}_{BOOT} \approx MSE$

1 Nonparametric Bootstrap

$\underset{\sim}{y} = (y_1, \dots, y_n)$

Let $Y_1, \dots, Y_n \sim F$ with pdf $f(y)$. Recall, the empirical cdf is defined as

$$F_n(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(y_i \le y) \quad y \in \mathbb{R}.$$

MLE of F and as $n \to \infty$, $F_n \to F$

Theoretical: Sample $\underset{\sim}{y} \sim F$, use $y_1, \dots, y_n$ to compute F_n

Bootstrap: Sample $\underset{\sim}{y}^* \sim F_n$, use $y_1^*, \dots, y_n^*$ to compute F_n^*

of size n

The idea behind the nonparametric bootstrap is to sample many data sets from $F_n(y)$, which can be achieved by resampling from the data **with replacement.**

How many possible bootstrap samples? n^n

Are $y_1^*, \dots, y_n^*$ independent?

$$P(y_1^* = a, y_2^* = b) = \frac{\sum_{i=1}^{n} \mathbb{I}(y_i^* = a)}{n} \cdot \frac{\sum_{i=1}^{n} \mathbb{I}(y_2^* = b)}{n} = P(y_1^* = a)\, P(y_2^* = b) \Rightarrow \text{yes}$$

Do we always want this?

No! More later

```
# observed data
x <- c(2, 2, 1, 1, 5, 4, 4, 3, 1, 2)

# create 10 bootstrap samples
x_star <- matrix(NA, nrow = length(x), ncol = 10)
for(i in 1:10) {
  x_star[, i] <- sample(x, length(x), replace = TRUE)
}
x_star
```

```
##       [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
##  [1,]    1    2    4    1    2    1    2    3    3     4
##  [2,]    4    4    1    1    1    2    2    1    2     1
##  [3,]    2    2    2    4    5    4    4    5    1     4
##  [4,]    4    4    2    5    2    4    5    5    1     3
##  [5,]    2    1    5    1    3    2    4    2    4     4
##  [6,]    4    4    2    1    4    4    4    3    1     2
##  [7,]    1    1    2    1    2    1    2    2    3     1
##  [8,]    4    4    1    3    3    3    5    1    2     4
##  [9,]    4    1    2    3    2    1    2    1    4     2
## [10,]    3    4    5    1    5    4    5    2    4     1
```

```
# compare mean of the samp to the means of the bootstrap samples
mean(x)
```

$\hat{\theta}$

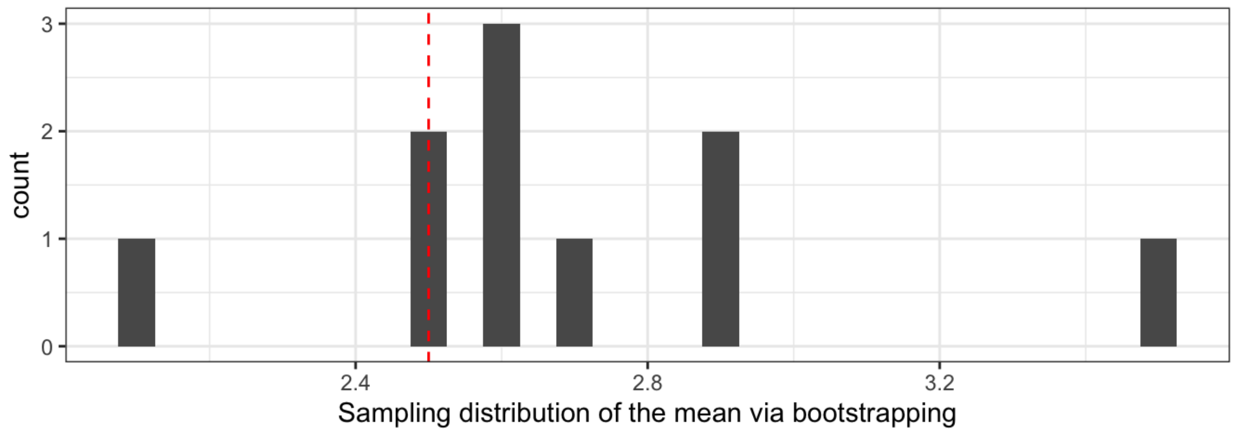
```
## [1] 2.5
```

```
colMeans(x_star)
```

$\hat{\theta}^{*(1)}, \dots, \hat{\theta}^{*(10)}$

```
##  [1] 2.9 2.7 2.6 2.1 2.9 2.6 3.5 2.5 2.5 2.6
```

```
ggplot() +
  geom_histogram(aes(colMeans(x_star)), binwidth = .05) +
  geom_vline(aes(xintercept = mean(x)), lty = 2, colour = "red") +
  xlab("Sampling distribution of the mean via bootstrapping")
```



1.1 Algorithm of NP bootstrap for iid data.

Goal: estimate the sampling distribution of a statistic based on observed data $y_1, \ldots, y_n$.

Let θ be the parameter of interest and $\hat{\theta}$ be an estimator of θ. Then,

For $b = 1, \ldots, B$

① Sample $y^{*(b)} = (y_1^{*(b)}, \ldots, y_n^{*(b)})$ by sampling w/ replacement from the sample data (i.e. sample from F_n)

② Compute $\hat{\theta}^{*(b)} = \hat{\theta}(y^{*(b)})$

estimate of θ based on b^{th} bootstrap sample.

Using $\hat{\theta}^{*(1)}, \ldots, \hat{\theta}^{*(B)}$ we can

- estimate the sampling dsn of $\hat{\theta}$ (histogram of $\hat{\theta}^{*(1)}, \ldots, \hat{\theta}^{*(B)}$).
- estimate the SE of $\hat{\theta}$ (st. dev. of $\hat{\theta}^{*(1)}, \ldots, \hat{\theta}^{*(B)}$).
- estimate a CI (many ways).

etc.

1.2 Justification for iid data

Suppose $Y_1, \dots, Y_n$ are iid with $\mathrm{E}Y_1 = \mu \in \mathbb{R}$, $\mathrm{Var}(Y_1) = \sigma^2 \in (0, \infty)$. Let's approximate the distribution of $T_n = \sqrt{n}(\bar{Y}_n - \mu)$ via the bootstrap.

Theorem: If $Y_1, Y_2, \dots$ are iid with $\mathrm{Var}(Y_1) = \sigma^2 \in (0, \infty)$, then $\sup_{y \in \mathbb{R}} |P(T_n \le y) - P_*(T_n^* \le y)| \equiv \Delta_n \to 0$ as $n \to \infty$ almost surely (a.s).

Given $\underline{Y} = \{Y_1, \dots, Y_n\}$ draw $Y_1^*, \dots, Y_n^*$ bootstrap sample. Then,

bootstrap probability. $\rightarrow P_*(Y_i^* = Y_i) = P(Y_i^* = Y_i \mid \underline{Y}) = \frac{1}{n} \quad 1 \le i \le n$

The bootstrap version of T_n is $T_n^* = \sqrt{n}(\bar{Y}_n^* - E_* Y_1^*) = \sqrt{n}(\bar{Y}_n^* - \bar{Y}_n)$

where $E_*[Y_i^*] = E[Y_i^* \mid \underline{Y}] = \sum_{i=1}^n \frac{1}{n} Y_i = \bar{Y}_n$ also $E_*(\bar{Y}_n^*) = E_*\left(\frac{1}{n}\sum_{i=1}^n Y_i^*\right) = \frac{1}{n}\sum_{i=1}^n E_* Y_i^* = \bar{Y}_n$

bootstrap expected value

exists b/c dsn of $Y_1^*, \dots, Y_n^*$ exist, but hard to compute directly b/c n^n bootstrap samples $\Rightarrow$ use simulation to estimate

Also, $P_*(T_n^* \le y) = P(T_n^* \le y \mid \underline{Y})$ approximates $P(T_n \le y)$, $y \in \mathbb{R}$ (Theorem).

The proof of this theorem requires two facts:

i. (Berry-Esseen Lemma) Let $Y_1, \dots, Y_n$ be independent with $\mathrm{E}Y_i = 0$ and $\mathrm{E}|Y_i|^3 < \infty$ for $i = 1, \dots, n$. Let $\sigma_n^2 = n\mathrm{Var}(\bar{Y}_n) = n^{-1}\sum_{i=1}^n \mathrm{E}Y_i^2 > 0$. Then,

$$\sup_{y \in \mathbb{R}} \left| P\left(\frac{\sqrt{n}\bar{Y}_n}{\sigma_n} \le y\right) - \Phi(y) \right| = \sup_{x \in \mathbb{R}} \left| P\left(\sqrt{n}\bar{Y}_n \le x\right) - \Phi\left(\frac{x}{\sigma_n}\right) \right| \le \frac{2.75}{n^{3/2}\sigma_n^3} \sum_{i=1}^n \mathrm{E}|Y_i|^3.$$

M-Z

ii. (Marcinkiewicz-Zygmund SLLN) Let X_i be a sequence of iid random variables with $\mathrm{E}|X_i|^p < \infty$ for $p \in (0, 2)$. Then, for $S_n = \sum_{i=1}^n X_i$,

$$\frac{1}{n^{1/p}}(S_n - nc) \to 0 \text{ as } n \to \infty \text{ almost surely } (*)$$

for any $c \in \mathbb{R}$ if $p \in (0, 1)$ and for $c = \mathrm{E}X_1$ if $p \in [1, 2)$. If $(*)$ holds for some $c \in \mathbb{R}$, then $\mathrm{E}|X_1|^p < \infty$.

Specifically, we will use that if $\{Y_i\}$ are iid w/ $EY_1^2 < \infty$, then

$$\frac{1}{n^{3/2}}\sum_{i=1}^n |Y_i|^3 \to 0 \quad \text{as} \quad n \to \infty \quad \text{a.s.}$$

letting $X_i = |Y_i|^3$ because $E|X_1|^p = E|Y_1|^{3p} < \infty$ for $p = 2/3$ we may take $c = 0$.

Proof:

Write $\sup_{y\in\mathbb{R}} |P(T_n \le y) - P_*(T_n^* \le y)| \le \underbrace{\sup_{y\in\mathbb{R}} |P(T_n \le y) - \Phi(y/\sigma)|}_{\tilde{\Delta}_n} + \underbrace{\sup_{y\in\mathbb{R}} |P_*(T_n^* \le y) - \Phi(y/\sigma)|}_{\Delta_{n1}}$

$\tilde{\Delta}_n \to 0$ by CLT since $Y_1, \ldots, Y_n$ iid, $EY_1^2 < \infty$.

Note that

$$\sigma_{n*}^2 \equiv n\,Var_*(\bar{Y}_n^*) = n\,Var_*\left(\frac{1}{n}\sum_{i=1}^n Y_i^*\right) = \frac{n}{n^2}\sum_{i=1}^n Var_* Y_i^* = Var_* Y_1^*$$

$$= E_*(Y_1^*)^2 - [E_* Y_1^*]^2 \quad \text{where} \quad Y_1^* = \begin{cases} Y_1 & \text{w.p. } \frac{1}{n} \\ \vdots & \\ Y_n & \text{w.p. } \frac{1}{n} \end{cases}$$

$$= \frac{1}{n}\sum_{i=1}^n Y_i^2 - [\bar{Y}_n]^2$$

So, $\sigma_{n*}^2 \to EY_1^2 - (EY_1)^2 = \sigma^2$ as $n\to\infty$ w.p. by SLLN since $EY_1^2 < \infty$.

By the Berry Esseen Lemma on $T_n^* = \sqrt{n}(\bar{Y}_n^* - E_* Y_1^*)$ and $|a-b| \le 2\max\{|a|,|b|\}$
$\Rightarrow |a-b|^3 \le 8\max\{|a|^3, |b|^3\}$
$\le 8(|a|^3 + |b|^3)$

$$\sup_{y\in\mathbb{R}} \left|P_*(T_n^* \le y) - \Phi\left(\frac{y}{\sigma_{n*}}\right)\right| \overset{\text{Berry Esseen}}{\le} \frac{2.75}{n^{3/2}\sigma_{n*}^3}\, n\, \underbrace{E_*|Y_1^* - E_* Y_1^*|^3}_{= \sum_{i=1}^n \frac{1}{n}|Y_i - \bar{Y}_n|^3}$$

$T_n^* = \sqrt{n}(\bar{Y}_n^* - E_* Y_1^*)$

$$= \frac{2.75}{n^{1/2}\sigma_{n*}^3}\,\frac{1}{n}\sum_{i=1}^n |Y_i - \bar{Y}_n|^3$$

$$\le \frac{2.75}{n^{1/2}\sigma_{n*}^3}\,\frac{1}{n}\sum_{i=1}^n 8(|Y_i|^3 + |\bar{Y}_n|^3)$$

$$= \frac{8(2.75)}{\sigma_{n*}^3}\left(\boxed{\frac{|\bar{Y}_n|^3}{n^{1/2}}} + \underbrace{\frac{1}{n^{3/2}}\sum_{i=1}^n |Y_i|^3}\right)$$

$\to 0$ as $n\to\infty$ w.p. 1 since $\bar{Y}_n \to \mu$ w.p. 1.

$\to 0$ as $n\to\infty$ w.p. 1 by M-Z SLLN (prev. page).

Finally, use $\sigma_{n*}^2 \to \sigma^2$ w.p. 1 $\Rightarrow \Phi\left(\frac{y}{\sigma_{n*}}\right) \to \Phi\left(\frac{y}{\sigma}\right)$ as $n\to\infty$ for any $y\in\mathbb{R}$ since $\Phi(\cdot)$ is continuous.

Thus $\sup_{y\in\mathbb{R}} \left|\Phi\left(\frac{y}{\sigma_{n*}}\right) - \Phi\left(\frac{y}{\sigma}\right)\right| \to 0$ as $n\to\infty$ w.p. 1 (Polya's Theorem).

So, $\Delta_{n1} = \sup_{y\in\mathbb{R}} \left|P_*(T_n^* \le y) - \Phi\left(\frac{y}{\sigma}\right)\right| \le \sup_{y\in\mathbb{R}} \left|P_*(T_n^* \le y) - \Phi\left(\frac{y}{\sigma_{n*}}\right)\right| + \sup_{y\in\mathbb{R}} \left|\Phi\left(\frac{y}{\sigma_{n*}}\right) - \Phi\left(\frac{y}{\sigma}\right)\right| \to 0$ wp 1 as $n\to\infty$. //

1.3 Properties of Estimators

We can use the bootstrap to estimate different properties of estimators.

1.3.1 Standard Error

Recall $se(\hat{\theta}) = \sqrt{Var(\hat{\theta})}$. We can get a **bootstrap** estimate of the standard error:

$$\hat{se}(\hat{\theta}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\hat{\theta}^{*(b)} - \bar{\hat{\theta}}^{*}\right)^2} \quad \text{where} \quad \bar{\hat{\theta}}^{*} = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}^{(b)}$$

1.3.2 Bias

Recall bias$(\hat{\theta}) = E[\hat{\theta} - \theta] = E[\hat{\theta}] - \theta$. We can get a **bootstrap** estimate of the bias:

$$\hat{\text{bias}}(\hat{\theta}) = \underbrace{\bar{\hat{\theta}}^{*}}_{\text{computed from Bootstrap samples}} - \underbrace{\hat{\theta}}_{\text{based on original data}}$$

Overall, we seek statistics with $\underbrace{\text{small se and small bias.}}_{}$

$$MSE = \text{Variance} + \text{Bias}^2 = E\left[(\hat{\theta} - \theta)^2\right].$$

$\Rightarrow$ Bootstrap procedure to estimate the MSE:

1. Compute $\hat{\theta}$ from original data $Y = (Y_1, \ldots, Y_n)$
2. Take B bootstrap samples of size n from data $\underline{Y}^{*(1)}, \ldots, \underline{Y}^{*(B)}$
3. Compute $\hat{\theta}^{*(b)}$ estimate of θ obtained from b^{th} BS sample
4. $\hat{MSE} = \frac{1}{B}\sum_{b=1}^{B}\left(\hat{\theta}^{*(b)} - \hat{\theta}\right)^2$

1.4 Sample Size and # Bootstrap Samples

$$n = \text{sample size} \quad \& \quad B = \# \text{ bootstap samples}$$

If n is too small, or sample isn't representative of the population,

bootstrap results will be poor no matter how large B is.

Guidelines for B –

$B \approx 1000$ for estimating bias, se

$B \approx 2000$ for CI (depends on α: small $\alpha \Rightarrow \uparrow B$)

Best approach –

Repeat bootstrap w/ different seeds. If estimates very different, $\uparrow B$

Your Turn

In this example, we explore bootstrapping in the rare case where we know the values for the entire population. If you have all the data from the population, you don't need to bootstrap (or really, inference). It is useful to learn about bootstrapping by comparing to the truth in this example.

In the package `bootstrap` is contained the average LSAT and GPA for admission to the population of 82 USA Law schools (an old data set – there are now over 200 law schools). This package also contains a random sample of size $n = 15$ from this dataset.

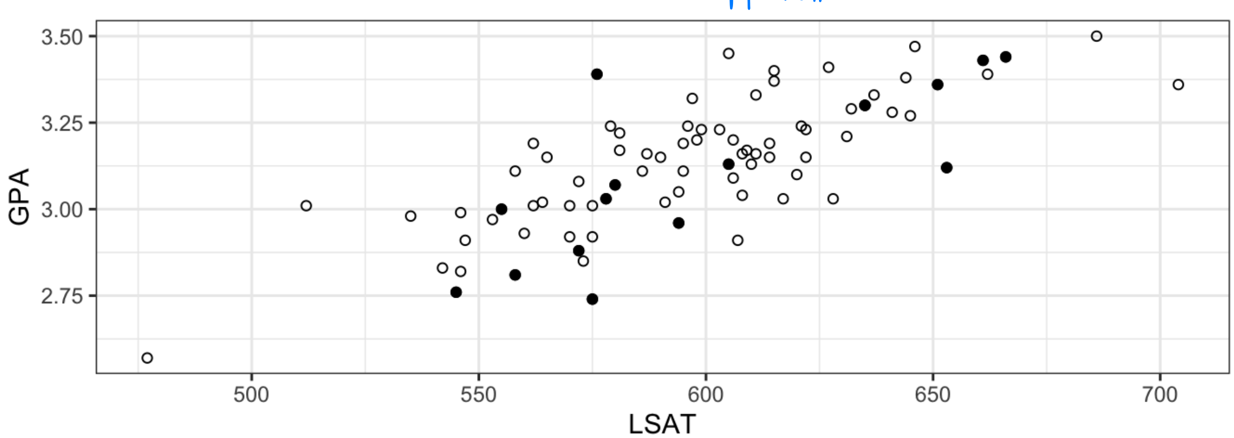
```
library(bootstrap)

head(law)
```

random sample of size $n=15$

```
##   LSAT  GPA
## 1  576 3.39
## 2  635 3.30
## 3  558 2.81
## 4  578 3.03
## 5  666 3.44
## 6  580 3.07
```

```
ggplot() +
  geom_point(aes(LSAT, GPA), data = law) +
  geom_point(aes(LSAT, GPA), data = law82, pch = 1)
```

full population.



correlation

We will estimate the correlation $\theta = \rho(\text{LSAT}, \text{GPA})$ between these two variables and use a bootstrap to estimate the sample distribution of $\hat{\theta}$.

```
# sample correlation
cor(law$LSAT, law$GPA)
```

```
## [1] 0.7763745
```

```
# population correlation
cor(law82$LSAT, law82$GPA)
```

```
## [1] 0.7599979
```

```
# set up the bootstrap
B <- 200
n <- nrow(law)
r <- numeric(B) # storage for replicates.

for(b in B) {
  ## Your Turn: Do the bootstrap!
}
```

1. Plot the sample distribution of $\hat{\theta}$. Add vertical lines for the true value θ and the sample estimate $\hat{\theta}$.

2. Estimate $sd(\hat{\theta})$.

3. Estimate the bias of $\hat{\theta}$

1.5 Bootstrap CIs

We will look at ~~five~~ four different ways to create confidence intervals using the boostrap and discuss which to use when.

1. Percentile Bootstrap CI
2. Basic Bootstrap CI
3. ~~Standard Normal Bootstrap CI~~
4. Bootstrap t
5. Accelerated Bias-Corrected (BCa)

   adjusted for skewness

Key ideas:

(1) When you say used "bootstrapping" to estimate a CI, need to say which one.

(2) (For now) whatever you are bootstrapping needs to be independent.

(3) Bootstrapping is an attempt to simulate replication. (think about interpretation of CI).

1.5.1 Percentile Bootstrap CI (Probably the one you're thinking of).

Let $\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}$ be bootstrap replicates and let $\hat{\theta}_{\alpha/2}$ be the $\alpha/2$ quantile of $\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}$.

Then, the $100(1-\alpha)\%$ two-sided Percentile Bootstrap CI for θ is

$$(\hat{\theta}_{\alpha/2}, \hat{\theta}_{1-\alpha/2})$$

↑ if bootstrap dsn is reasonably approximated by a ctsˌ dsn doesn't matter if we include end points.

In R, if `bootstrap.reps = c(`$\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}$`)`, the percentile CI is
vector of bootstrap statistics.

```
quantile(bootstrap.reps, c(alpha/2, 1 - alpha/2))
```

Assumptions/usage

- Widely used because its simple to implement and explain.
- drawback: CI's usually too narrow, leading to low coverage.
  ↳ more often when bias or skewness in bootstrap dsn.

Justification (Efron):

Assume the existence of an increasing transformation g s.t. $P[g(\hat{\theta}) - g(\theta) \le x] = H(x)$ ← dsn function of a r.v. symmetric about 0, i.e. $H^{-1}(\alpha) = -H^{-1}(1-\alpha)$ for $0<\alpha<1$

$\Rightarrow$ "bootstrap world": $P_*[g(\hat{\theta}^*) - g(\hat{\theta}) \le x] \approx H(x)$. $(*)$.

Suppose g is known, then $P[g^{-1}(g(\hat{\theta}) - x) \le \theta] = H(x)$. Then set this probability to $1-\alpha$.

$P[g^{-1}(g(\hat{\theta}) - H^{-1}(1-\alpha)) \le \theta] = 1-\alpha \Rightarrow$ if g is known this is a one-sided CI w/ coverage prob $1-\alpha$

$\left(\underline{g^{-1}(g(\hat{\theta}) - H^{-1}(1-\alpha))}, \infty\right)$. (upper bound version similar)

Goal: Show $g^{-1}(g(\hat{\theta}) - H^{-1}(1-\alpha))$ is estimated by the left endpoint of a percentile bootstrap interval $\hat{\theta}_\alpha$

$$\alpha = P_*(\hat{\theta}^* \le \hat{\theta}_\alpha) = P_*(g(\hat{\theta}^*) \le g(\hat{\theta}_\alpha))$$

$$= P_*(g(\hat{\theta}^*) - g(\hat{\theta}) \le g(\hat{\theta}_\alpha) - g(\hat{\theta})) \approx H(g(\hat{\theta}_\alpha) - g(\hat{\theta})) \text{ by } (*).$$

$$H^{-1}(\alpha) \approx g(\hat{\theta}_\alpha) - g(\hat{\theta}) \Rightarrow g^{-1}(\underbrace{H^{-1}(\alpha)}_{-H^{-1}(1-\alpha)} + g(\hat{\theta})) \approx \hat{\theta}_\alpha$$

$$= \underline{g^{-1}(g(\hat{\theta}) - H^{-1}(1-\alpha))}$$

NOTE: we do not need to know g to calculate $\hat{\theta}_\alpha$!

1.5.2 Basic Bootstrap CI (corrects for bias)

based on residuals.

The $100(1-\alpha)\%$ Basic Bootstrap CI for θ is

2-sided

$$\left(\hat{\theta} - [\hat{\theta}_{1-\alpha/2} - \hat{\theta}], \hat{\theta} - [\hat{\theta}_{\alpha/2} - \hat{\theta}]\right) = \left(2\hat{\theta} - \hat{\theta}_{1-\alpha/2}, 2\hat{\theta} - \hat{\theta}_{\alpha/2}\right).$$

estimate from original data

$1-\alpha/2$ quantile of bootstrap distribution

why? let $\varepsilon = \hat{\theta} - \theta$ and $\varepsilon_{1-\alpha/2}, \varepsilon_{\alpha/2}$ quantiles of the distribution of ε.

$$P(\varepsilon_{\alpha/2} < \hat{\theta} - \theta < \varepsilon_{1-\alpha/2}) = 1-\alpha \implies (1-\alpha)100\% \text{ CI is } \left(\hat{\theta} - \varepsilon^*_{1-\alpha/2}, \hat{\theta} - \varepsilon^*_{\alpha/2}\right)$$

where $\varepsilon^*_{\alpha/2}$ is quantile of the bootstrap distribution of ε^*

Assumptions/usage

- corrects for bias, but slightly harder to explain.

misnomer

1.5.3 Bootstrap t CI (Studentized Bootstrap)

Consider $z = \frac{\hat{\theta} - E(\hat{\theta})}{se(\hat{\theta})}$.

Even if the distribution of $\hat{\theta}$ is Normal and $\hat{\theta}$ is unbiased for θ, the Normal distribution is not exactly correct for z.

because we need to estimate $se(\hat{\theta}) \Rightarrow t^* = \frac{\hat{\theta} - E(\hat{\theta})}{\hat{se}(\hat{\theta})} \sim t_{n-1}$? NO

where $\hat{se}(\hat{\theta}) = sd(\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(B)})$

Additionally, the distribution of $\hat{se}(\hat{\theta})$ is unknown.

so we cannot claim $t^* \sim t_{n-1}$ even when $\hat{\theta} \sim$ Normal!

$\Rightarrow$ The bootstrap t interval does not use a Student t distribution as the reference distribuion, instead we estimate the distribution of a "t type" statistic by resampling.

The $100(1-\alpha)\%$ Boostrap t CI is

$$\left(\hat{\theta} - t^*_{1-\alpha/2} \cdot \hat{se}(\hat{\theta}),\ \hat{\theta} + t^*_{\alpha/2}\hat{se}(\hat{\theta})\right)$$

= quantile of the bootstrap "t-type" statistic.

estimated based on bootstrap samples $\hat{\theta}^{*(1)}, \ldots, \hat{\theta}^{(B)}$

Overview

t-type statistic: $t^{(1)} = \frac{\hat{\theta}^{(1)} - \hat{\theta}}{\underbrace{\hat{se}(\hat{\theta}^{(1)})}_{}}, \ldots, t^{(B)} = \frac{\hat{\theta}^{(B)} - \hat{\theta}}{\hat{se}(\hat{\theta}^{(B)})}$

= bootstrap estimate of $se(\hat{\theta})$ based on the 1st bootstrap sample ?? DOUBLE BOOTSTRAP!

To estimate the "t style distribution" for $\hat{\theta}$,

1. Compute $\hat{\theta}$
2. For each replicate $b = 1, \ldots, B$
   (a) Sample w/ replacement from y, $y^{(b)} = (y_1^{(b)}, \ldots, y_n^{(b)})$
   (b) Compute $\hat{\theta}^{(b)}$
   (c) For each replicate $r = 1, \ldots, R$
      i. sample w/ replacement from $y^{(b)}$, $y^{(b)(r)} = (y_1^{(b)(r)}, \ldots, y_n^{(b)(r)})$
      ii. Compute $\hat{\theta}^{(b)(r)}$
   (d) Compute $\hat{se}(\hat{\theta}^{(b)}) = sd(\hat{\theta}^{(b)(1)}, \ldots, \hat{\theta}^{(b)(R)})$
   (e) Compute t-style statistic $t^{(b)} = \frac{\hat{\theta}^{(b)} - \hat{\theta}}{\hat{se}(\hat{\theta}^{(b)})}$.
3. get quantiles $t^*_{\alpha/2}, t^*_{1-\alpha/2}$
4. Compute CI.

Assumptions/usage

- Computationally intensive
- Not doing anything for bias/skewness in bootstrap dsn.
- Need $\hat{\theta}$ independent of $\hat{se}(\hat{\theta})$.

This idea is based on "pivotal quantities" = a function of observations and parameters whose probability dsn does not depend on the parameters.

e.g. $\dfrac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}$ doesn't depend on μ.

You could create other pivotal quantities, e.g.

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{if} \quad Y_i \overset{iid}{\sim} N(\mu, \sigma^2).$$

$$\Rightarrow \left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2}} \right) \text{ would be a } (1-\alpha)100\% \text{ CI for } \sigma^2.$$

Bootstrap version of this, do not assume $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$.

We would now need to estimate a χ^2-type quantile.

"Bias-corrected accelerated"

1.5.4 BCa CIs

Modified version of percentile intervals that adjusts for bias of estimator and skewness of the sampling distribution.

This method automatically selects a transformation so that the normality assumption holds.

Idea:

Assume there exist a monotonically increasing function g and constants a, b s.t.

$$\frac{g(\hat\theta)-g(\theta)}{1+ag(\theta)} + b \sim N(0,1) \quad \text{where} \quad 1+ag(\theta) > 0.$$

Let Z_p be the $100p^{th}$ percentile of the standard Normal $\xRightarrow{\text{to get CI}}$ $-Z_p < \frac{g(\hat\theta)-g(\theta)}{1+ag(\theta)} + b < Z_p$

$$\Leftrightarrow \frac{g(\hat\theta)+b-Z_p}{1-a(b-Z_p)} < g(\theta) < \frac{g(\hat\theta)+b+Z_p}{1-a(b+Z_p)}$$

is a $100(1-\alpha)\%$ CI if $p = 1-\alpha/2$.

Bootstrap version: $\frac{g(\hat\theta^*)-g(\hat\theta)}{1+ag(\hat\theta)} + b \overset{\text{approx}}{\sim} N(0,1)$.

$$\Rightarrow P\left(g(\hat\theta) < \frac{g(\hat\theta)+b+Z_p}{1-a(b+Z_p)}\right) = P\left(\underbrace{\frac{g(\hat\theta^*)-g(\hat\theta)}{1+ag(\hat\theta)}+b}_{\text{approx} \sim N(0,1).} < \frac{b+Z_p}{1-a(b+Z_p)} + b\right)$$

~~Thus the BCA interval for θ is~~ $(Z_L, Z_U) = \left(\frac{b-Z_p}{1-a(b-Z_p)} + b, \frac{b+Z_p}{1-a(b+Z_p)} + b\right)$.

Thus the BCA interval for θ has the same cumulative probability as these are different quantiles of $N(0,1)$.

e.g. if $b = 0.2$, $a = 0.01$, $\alpha = 0.05$, $Z_L = -1.53$, $Z_U = 2.41$

6.3^{th} percentile of $N(0,1)$ — 99.2^{th} percentile of $N(0,1)$.

$\Rightarrow \underbrace{(\hat\theta_{.063}, \hat\theta_{.992})}_{\text{BCA interval}}$ quantiles of the Bootstrap distribution.

How to find a and b? $\frac{g(\hat\theta^*)-g(\hat\theta)}{1+ag(\hat\theta)} \overset{\text{approx}}{\sim} N(b,1) \Rightarrow$

b corrects for bias.

Let p_0 be the fraction of observations from bootstrap dsn s.t. $\hat\theta^{*(i)} < \hat\theta$. Since g is monotone, this is the same fraction of $\hat\theta^*$'s s.t. $g(\hat\theta^*) < g(\hat\theta)$.

$\Rightarrow p_0 = P(Z < b)$ where $Z \sim N(0,1)$ gives a way to approximate b. (If the bootstrap dsn has $\hat\theta$ as its median, then $b = 0$)

a corrects for skewness

a: Let $S_{-i} = \{Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n\}$ and let $\hat\theta_{-i}$ denote the estimate of θ based on S_{-i}

Assume there exists a monotonically increasing function g and constants a and b s.t.

$$U = \frac{g(\hat{\theta}) - g(\theta)}{1 + a g(\theta)} + b \sim N(0,1) \text{ where } 1 + a g(\theta) \geq 0.$$

"Bootstrap World" $U^* = \dfrac{g(\hat{\theta}^*) - g(\hat{\theta})}{1 + a g(\hat{\theta})} + b \overset{\text{approx}}{\sim} N(0,1).$

$\Rightarrow$ For any quantile of a standard Normal,

$$\begin{aligned}
\alpha &\approx P^*\left[U^* \leq z_\alpha\right] \\
&= P^*\left[\frac{g(\hat{\theta}^*) - g(\hat{\theta})}{1 + a g(\hat{\theta})} + b \leq z_\alpha\right] \\
&= P^*\Big[\hat{\theta}^* \leq \underbrace{g^{-1}\left(g(\hat{\theta}) + (z_\alpha - b)(1 + a g(\hat{\theta}))\right)}_{\hat{\theta}_\alpha}\Big]
\end{aligned}$$

The α quantile from the bootstrap distribution of $\hat{\theta}^*$, denoted $\hat{\theta}_\alpha$ is observable from the bootstrap dsn.

$$\Rightarrow \underline{g^{-1}\left(g(\hat{\theta}) + (z_\alpha - b)(1 + a g(\hat{\theta}))\right)} \approx \hat{\theta}_\alpha.$$

To use this, consider U: $1 - \alpha = P(U > z_\alpha)$

$$= P\left[\theta < \underline{g^{-1}\left(g(\hat{\theta}) + \frac{b - z_\alpha}{1 - a(b - z_\alpha)}\left[1 + a g(\hat{\theta})\right]\right)}\right]$$

Noticing this $\boxed{\text{similarity}}$, if we could find β s.t. $\dfrac{b - z_\alpha}{1 - a(b - z_\alpha)} = z_\beta - b$ then the bootstrap principle can be applied to conclude $\theta < \hat{\theta}_\beta$ can be an approximate $1-\alpha$ upper CI limit.

$$z_\beta = \frac{b - z_\alpha}{1 - a(b - z_\alpha)} + b \Rightarrow \beta = \Phi\left(\frac{b + z_{1-\alpha}}{1 - a(b + z_{1-\alpha})} + b\right)$$

symmetry of $N(0,1)$.

β^{th} quantile of $N(0,1)$.

$\Rightarrow$
1. Find a, b using formulas from previous page.
2. Compute β
3. Find the β^{th} quantile of empirical dsn of $\hat{\theta}^*$

Two-sided is similar argument.

$$a = \underbrace{\frac{\sum_{i=1}^n (\hat\theta_{( )} - \hat\theta_{-i})^3}{6\left[\sum_{i=1}^n (\hat\theta_{( )} - \hat\theta_{-i})^2\right]^{3/2}}}_{\text{based on skewness estimate.}}$$ where $\hat\theta_{( )}$ is the mean of $\hat\theta_{-i}$'s.

If $a=0$, don't adjust for skewness $\Rightarrow$ BC interval.

The BCa method uses bootstrapping to estimate the bias ~~and, skewness~~ and estimate skewness then modifies which percentiles are chosen to get the appropriate confidence limits for a given data set.

In summary,

Has better coverage in empirical studies, but harder to explain.

BCa is like the percentile bootstrap CI, but instead of $(\hat\theta_{\alpha/2}, \hat\theta_{1-\alpha/2})$, choose "better" quantiles to account for bias and skewness.

Your Turn

We will consider a telephone repair example from Hesterberg (2014). `Verizon` has repair times, with two groups, CLEC and ILEC, customers of the "Competitive" and "Incumbent" local exchange carrier.

other carriers customers

Verizon customers

Verizon required by law to serve both at the same speed.

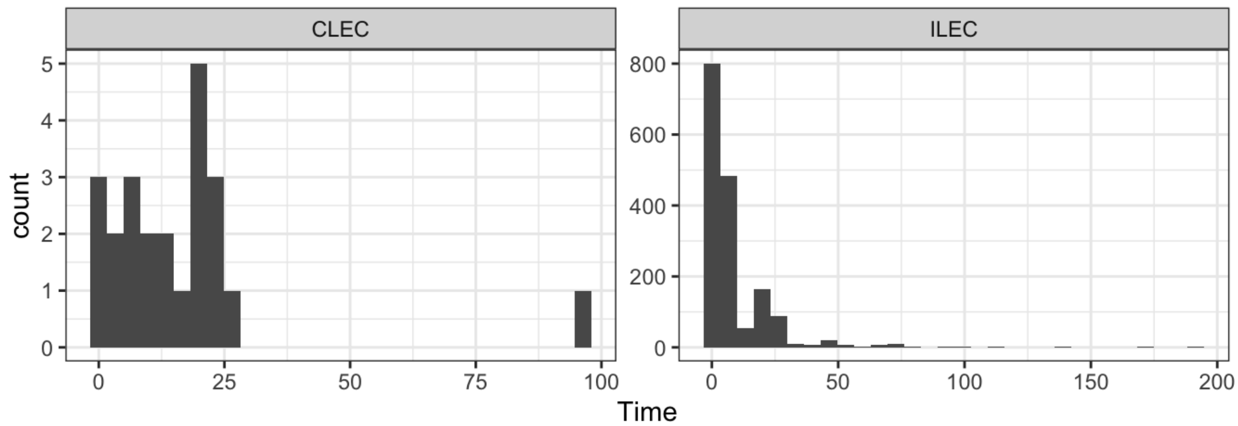
```
library(resample) # package containing the data

data(Verizon)
head(Verizon)
```

```
##    Time Group
## 1 17.50  ILEC
## 2  2.40  ILEC
## 3  0.00  ILEC
## 4  0.65  ILEC
## 5 22.23  ILEC
## 6  1.20  ILEC
```

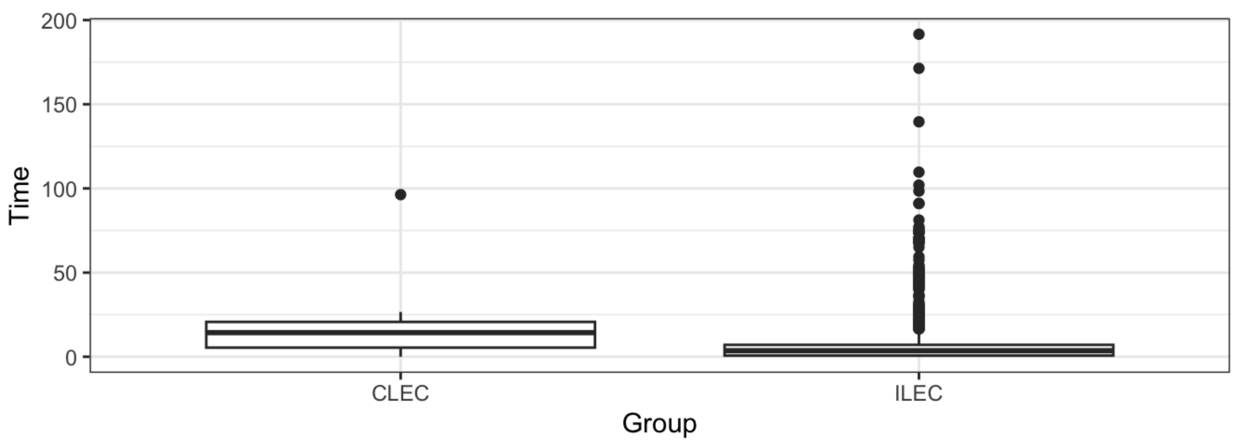
```
Verizon |>
  group_by(Group) |>
  summarize(mean = mean(Time), sd = sd(Time), min = min(Time), max =
         max(Time)) |>
  kable()
```

Group	mean	sd	min	max	n
CLEC	16.509130	19.50358	0	96.32	23
ILEC	8.411611	14.69004	0	191.60	1664

```
ggplot(Verizon) +
  geom_histogram(aes(Time)) +
  facet_wrap(.~Group, scales = "free")
```




```r
ggplot(Verizon) +
  geom_boxplot(aes(Group, Time))
```



1.6 Bootstrapping CIs

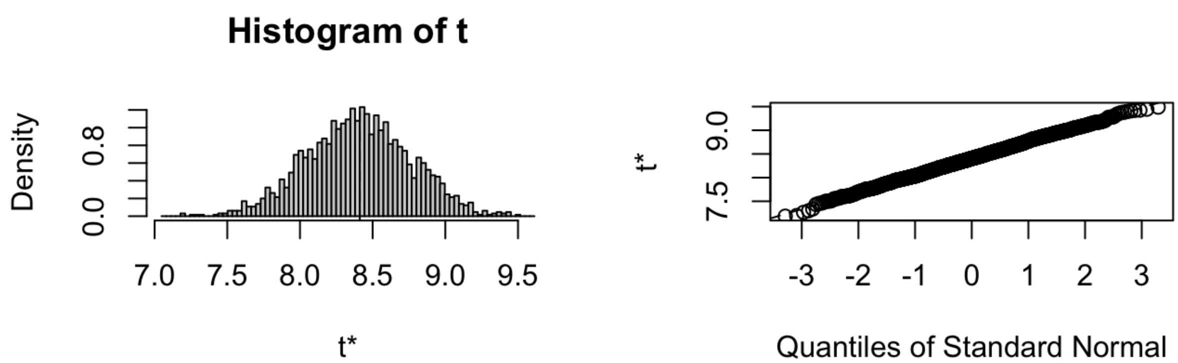
or just write your own functions. Unless I say otherwise, both are fine.

There are many bootstrapping packages in `R`, we will use the `boot` package. The function `boot` generates R resamples of the data and computes the desired statistic(s) for each sample. This function requires 3 arguments:

1. data = the data from the original sample (data.frame or matrix).
2. statistic = a function to compute the statistic from the data where the first argument is the data and the second argument is the indices of the obervations in the boostrap sample.
3. R = the number of bootstrap replicates.

```r
library(boot) # package containing the bootstrap function

mean_func <- function(x, idx) {
  mean(x[idx])
}

ilec_times <- Verizon[Verizon$Group == "ILEC",]$Time
boot.ilec <- boot(ilec_times, mean_func, 2000)

plot(boot.ilec)
```

bootstrap object.



If we want to get Bootstrap CIs, we can use the `boot.ci` function to generate the different nonparametric bootstrap confidence intervals.

```r
boot.ci(boot.ilec, conf = .95, type = c("perc", "basic", "bca"))
```

```
## BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
## Based on 2000 bootstrap replicates
##
## CALL :
## boot.ci(boot.out = boot.ilec, conf = 0.95, type = c("perc",
"basic",
##     "bca"))
##
## Intervals :
## Level      Basic              Percentile            BCa
## 95%   ( 7.733,  9.110 )   ( 7.714,  9.091 )   ( 7.755,  9.125 )
## Calculations and Intervals on Original Scale
```

```
## we can do some of these on our own
## percentile
quantile(boot.ilec$t, c(.025, .975))
```

```
##     2.5%    97.5%
## 7.714075 9.084725
```

very similar
not biased.

```
## basic
2*mean(ilec_times) - quantile(boot.ilec$t, c(.975, .025))
```

```
##    97.5%     2.5%
## 7.738496 9.109147
```

To get the studentized bootstrap CI, we need our statistic function to also return the variance of $\hat{\theta}$.

```
mean_var_func <- function(x, idx) {
  c(mean(x[idx]), var(x[idx])/length(idx))
}

boot.ilec_2 <- boot(ilec_times, mean_var_func, 2000)
boot.ci(boot.ilec_2, conf = .95, type = "stud")
```

```
## BOOTSTRAP CONFIDENCE INTERVAL CALCULATIONS
## Based on 2000 bootstrap replicates
##
## CALL :
## boot.ci(boot.out = boot.ilec_2, conf = 0.95, type = "stud")
##
## Intervals :
## Level    Studentized
## 95%   ( 7.728,  9.183 )
## Calculations and Intervals on Original Scale
```

Which CI should we use?

All very similar.

Maybe percentile if needing to explain to a judge/jury.

Maybe BCA b/c has shown better coverage.

1.7 Bootstrapping for the difference of two means

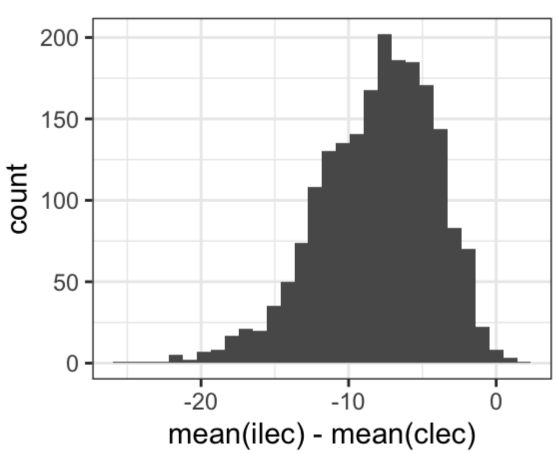
Given iid draws of size n and m from two populations, to compare the means of the two groups using the bootstrap,

1. For replicates $b = 1, \dots, B$
   a) Draw a sample of size n w/ replacement from sample 1 and separately a sample of size m w/ replacement from sample 2.
   b) Compute a statistic that compares the groups (i.e. $\hat{\theta} = \bar{y}_1 - \bar{y}_2$).
2. Have a bootstrap dsn of $\hat{\theta}^{(1)}, \dots, \hat{\theta}^{(B)}$ — shape, bias, se, etc.
3. Compute an appropriate CI.

again feel free to write your own.

The function `two.boot` in the `simpleboot` package is used to bootstrap the difference between univariate statistics. Use the bootstrap to compute the shape, bias, and bootstrap sample error for the samples from the `Verizon` data set of CLEC and ILEC customers.

```
library(simpleboot)

clec_times <- Verizon[Verizon$Group == "CLEC",]$Time
diff_means.boot <- two.boot(ilec_times, clec_times, "mean", R = 2000)

ggplot() +
  geom_histogram(aes(diff_means.boot$t)) +
  xlab("mean(ilec) - mean(clec)")
```



looks skewed.

```
# Your turn: estimate the bias and se of the sampling distribution
```

Which confidence intervals should we use?

```
# Your turn: get the chosen CI using boot.ci
```

Is there evidence that

$$
\begin{aligned}
H_0 &: \mu_1 - \mu_2 = 0 \\
H_a &: \mu_1 - \mu_2 < 0
\end{aligned}
$$

is rejected?